Published by Institute of Physics Publishing for SISSA/ISAS
Received: *August 28, 2002*
Accepted: *September 11, 2002*

Higher level open string states from vacuum string field theory

Hiroyuki Hata and Hisashi Kogetsu

Department of Physics, Kyoto University
Kyoto 606-8502, Japan
E-mail: hata@gauge.scphys.kyoto-u.ac.jp kogetsu@gauge.scphys.kyoto-u.ac.jp

Abstract: We construct massive open string states around a classical solution in the oscillator formulation of Vacuum String Field Theory. In order for the correct mass spectrum to be reproduced, the projection operators onto the modes of the left- and right-half of the string must have an anomalous eigenvalue 1/2, and the massive states are constructed using the corresponding eigenvector. We analyze numerically the projection operators by regularizing them to finite size matrices and confirm that they indeed have eigenvalue 1/2. Beside the desired massive states, we have spurious massive as well as massless states, which are infinitely degenerate. We show that these unwanted states can be gauged away.

Keywords: Bosonic Strings, Tachyon Condensation, String Field Theory.

JHEP09(2002)027

Contents

1. Introduction and summary

Vacuum String Field Theory (VSFT) [1]–[4] has been proposed as a string field theory expanded around the tachyon vacuum. In order for VSFT to really be connected to ordinary bosonic string theory on an unstable D25-brane, there must exist a Lorentz and translationally invariant classical solution of VSFT satisfying the following two requirements: the fluctuation modes around the solution reproduce the open string spectrum, and the energy density of the solution is equal to the D25-brane tension. Recently, there has been much progress in understanding the above problem. In particular, a full classical solution of VSFT including the ghost part has been presented in [5] using the oscillator formulation, and the tachyon and the massless vector fluctuation modes have been constructed there. They have shown that the tachyon mass is correctly reproduced. However, the massless vector mode contains an arbitrary vector in the level number space, implying that there are infinite number of massless vector states. This problem was later resolved by Imamura [10]: most of the massless vector modes can be gauged away by VSFT gauge transformation leaving only one physical vector mode.

The purpose of this paper is to carry out the oscillator construction of fluctuation modes representing higher level massive modes of open string.[1] This is in fact a non-trivial and interesting problem. Let us take, as a candidate massive state with mass squared equal to $(k-1)\alpha'$, a state given as k matter creation operators $a_n^{\mu\dagger}$ acting on the tachyon state. It is a natural extension of the tachyon and massless vector states of [5]. However, naive analysis shows that this kind of states are all massless. The wave equation (namely, the linearized equation of motion), $\mathcal{Q}_\mathrm{B}\Phi = 0$, for the fluctuation Φ of the above type is reduced

[1] In this paper we shall consider only the oscillator formulation of VSFT. For the construction of the fluctuation modes using boundary conformal field theory, see [6]–[9].

to a simple algebraic equation consisting only of the projection operators, ρ_+ and ρ_-, onto the modes of the left- and right-half of the string [11]. The masslessness is a consequence of the basic property of projection operators, $\rho_\pm^2 = \rho_\pm$.

We find, however, that the above mode can represent a massive state with the expected mass squared, $(k-1)\alpha'$, if $\rho_\pm$ has an anomalous eigenvalue $1/2$ despite that it is a projection operator. Such an anomalous eigenvalue is of course impossible for projection operators in a finite dimensional space. However, there is a subtle point for $\rho_\pm$ which is an operator in the infinite dimensional space of string level number. In fact, the eigenvector $\boldsymbol{f}^{(\kappa)}$ of the matrix representation of the Virasoro algebra $K_1 = L_1 + L_{-1}$ (the eigenvalue κ is continuous and extending from $-\infty$ to ∞) is at the same time the eigenvector of $\rho_\pm$ and the corresponding eigenvalue is the step function $\theta(\pm\kappa)$ [12]. Therefore, the vector $\boldsymbol{f}^{(0)}$ is the eigenvector of $\rho_\pm$ with eigenvalue $\theta(0)$, which is indefinite but could be the desired value $1/2$. To verify whether this expectation is correct, we have to study $\rho_\pm$ with some kind of regularization. In this paper, we analyze numerically the eigenvalue problem of $\rho_\pm$ regularized to finite size matrices to obtain results supporting the above expectation: $\rho_\pm$ has an eigenvalue and an eigenvector which tend to $1/2$ and $\boldsymbol{f}^{(0)}$, respectively, as the size of the matrices is increased.

Even if $\rho_\pm$ has the expected anomalous eigenvalue $1/2$, there still remains a problem to be solved for the construction of higher level open string modes. Analysis of the wave equation for fluctuations of the above type, $(a^\dagger)^k|\text{tachyon}\rangle$, shows that there still exists infinite degeneracy of massive states with mass squared equal to $\ell\alpha'$ ($\ell \leq k-2$). In addition, we also have spurious massless states mentioned above. We have to show that these unwanted states are not physical ones. This problem is solved in the same manner as in the massless vector case [10]: infinite number of spurious states can be gauged away. However, we need gauge transformations of a different kind from that used in [10] in order to remove all the unwanted massive states.

Our construction of massive open string states is not complete, and there remain a number of future problems. First, we have to present a rigorous analytic proof of the existence of the anomalous eigenvalue $1/2$ of the projection operators $\rho_\pm$. Second, as we shall see later, we construct only the highest spin states at a given mass level. The construction of lower spin states is our remaining subject. Finally, in our analysis we consider the wave equation $\mathcal{Q}_\mathrm{B}\Phi = 0$ only in the Fock space of first quantized string states. Namely, our massive modes Φ satisfy the wave equation in the sense of $\langle\text{Fock}|\mathcal{Q}_\mathrm{B}|\Phi\rangle = 0$ for any Fock space element $\langle\text{Fock}|$. However, analysis of the potential height problem of the D25-brane solution of VSFT shows that we have to consider the equation of motion in a larger space including the states constructed upon the D25-brane solution [6, 9, 13]. This is our important future problem.

The organization of the rest of this paper is as follows. In section 2, we analyze the equation of motion for our candidate massive modes and argue that $\rho_\pm$ needs an anomalous eigenvalue $1/2$. In section 3, we present numerical analysis of the eigenvalue problem of finite size $\rho_\pm$. In section 4, we show that the spurious states are unphysical ones which can be removed by gauge transformations. In appendix A, we present technical details used in the text.

JHEP09(2002)027

2. Massive modes

The action of VSFT is given by [1, 2, 4]

$$S = -K\left(\frac{1}{2}\Psi\cdot\mathcal{Q}\,\Psi + \frac{1}{3}\,\Psi\cdot(\Psi*\Psi)\right), \tag{2.1}$$

where K is a constant and the BRST operator $\mathcal{Q}$ of VSFT consists purely of ghost oscillators:

$$\mathcal{Q} = c_0 + \sum_{n=1}^{\infty} f_n\left(c_n + (-1)^n c_n^\dagger\right). \tag{2.2}$$

The VSFT action (2.1) is invariant under the gauge transformation,

$$\delta_\Lambda\Psi = \mathcal{Q}\Lambda + \Psi*\Lambda - \Lambda*\Psi\,. \tag{2.3}$$

The D25-brane configuration of VSFT is a translationally and Lorentz invariant solution Ψ_c to the equation of motion:

$$\mathcal{Q}\Psi_\mathrm{c} + \Psi_\mathrm{c}*\Psi_\mathrm{c} = 0\,. \tag{2.4}$$

Assuming that Ψ_c factorizes into the matter part $\Psi_\mathrm{c}^\mathrm{m}$ and the ghost one $\Psi_\mathrm{c}^\mathrm{g}$, $\Psi_\mathrm{c} = \Psi_\mathrm{c}^\mathrm{m}\otimes\Psi_\mathrm{c}^\mathrm{g}$, (2.4) is reduced into the following two:

$$\Psi_\mathrm{c}^\mathrm{m} = \Psi_\mathrm{c}^\mathrm{m}*\Psi_\mathrm{c}^\mathrm{m}\,, \tag{2.5}$$

$$\mathcal{Q}\Psi_\mathrm{c}^\mathrm{g} + \Psi_\mathrm{c}^\mathrm{g}*\Psi_\mathrm{c}^\mathrm{g} = 0\,. \tag{2.6}$$

Here we shall fix our convention for the Neumann coefficient matrices. The matter part of the three-string vertex defining the $*$-product is given in the oscillator representation by

$$|V^\mathrm{m}\rangle_{123} = \exp\left(-\sum_{r,s=1}^{3}\sum_{m,n\geq 0}\frac{1}{2}a_m^{(r)\dagger}V_{mn}^{rs}a_n^{(r)\dagger}\right)|p_1\rangle|p_2\rangle|p_3\rangle\,, \tag{2.7}$$

with $a_0 = \sqrt{2}\,p$ (we are taking the convention of $\alpha' = 1$). The Neumann coefficient matrices and the vectors, M_α and $\boldsymbol{v}_\alpha$ ($\alpha = 0, \pm$), are related to V^{rs} in (2.7) as follows:

$$\begin{aligned} &(M_0)_{mn} = (CV^{rr})_{mn}\,, \qquad &&(M_\pm)_{mn} = (CV^{r,r\pm1})_{mn}\,, \\ &(\boldsymbol{v}_0)_n = V_{n0}^{rr}\,, &&(\boldsymbol{v}_\pm)_n = V_{n0}^{r,r\pm1}\,, \qquad (m,n\geq 1) \end{aligned} \tag{2.8}$$

where C is the twist matrix, $C_{mn} = (-1)^m\delta_{mn}$.

The matter part solution $\Psi_\mathrm{c}^\mathrm{m}$ to (2.5) has been obtained as a squeezed state [14, 2]:

$$|\Psi_\mathrm{c}^\mathrm{m}\rangle = [\det(1-T\mathcal{M})]^{13}\exp\left(-\frac{1}{2}\sum_{m,n\geq 1}a_m^\dagger(CT)_{mn}a_n^\dagger\right)|0\rangle\,, \tag{2.9}$$

where the matrix T is given in terms of M_0 by

$$T = \frac{1}{2M_0}\left(1 + M_0 - \sqrt{(1-M_0)(1+3M_0)}\right). \tag{2.10}$$

JHEP09(2002)027

The ghost part solution Ψ_c^g to (2.6) has also been obtained by taking the Siegel gauge and assuming the squeezed state form [5]. Beside determining Ψ_c^g, (2.6) fixes the coefficients f_n in $\mathcal{Q}$ (2.2) which are arbitrary for the gauge invariance alone.

Let us express the VSFT field Ψ as a sum of Ψ_c and the fluctuation Φ:

$$\Psi = \Psi_c + \Phi \,. \tag{2.11}$$

Then the linear part of the equation of motion for Φ reads

$$\mathcal{Q}_B\Phi \equiv \mathcal{Q}\Phi + \Psi_c * \Phi + \Phi * \Psi_c = 0 \,, \tag{2.12}$$

where $\mathcal{Q}_B$ is the BRST operator around the classical solution Ψ_c. We would like to construct the fluctuation modes Φ corresponding to higher level open string states and satisfying the wave equation (2.12). We assume the factorization for these modes and that the ghost part is common to that of Ψ_c:

$$\Phi = \Phi^m \otimes \Psi_c^g \,. \tag{2.13}$$

Then the wave equation for the matter part Φ^m is given by

$$\Phi^m = \Psi_c^m * \Phi^m + \Phi^m * \Psi_c^m \,. \tag{2.14}$$

In the following we are interested only in the matter part Φ^m of the fluctuation modes and omit its superscript m. Equation (2.14) has been solved for the tachyon mode Φ_t [5]. Explicitly, it is given by

$$|\Phi_t\rangle = \exp\Biggl(-\sum_{n\geq 1} t_n a_n^\dagger a_0 + ip\,\widehat{x}\Biggr)|\Psi_c^m\rangle \,, \tag{2.15}$$

with

$$\boldsymbol{t} = 3(1+T)(1+3M_0)^{-1}\boldsymbol{v}_0 \,. \tag{2.16}$$

It has been shown that Φ_t (2.15) satisfies (2.14) when the momentum p_μ carried by Φ_t is on the tachyon mass-shell $p_\mu^2 = -m_{\text{tachyon}}^2 = 1$.

Now we shall start constructing fluctuation modes at a generic mass level. As a candidate fluctuation mode with mass squared equal to $k-1$, let us take the following one; k creation operators acting on the tachyon mode Φ_t:

$$|\Phi^{(k)}\rangle = \sum_{n_1,\cdots,n_k\geq 1} \beta_{n_1\cdots n_k}^{\mu_1\cdots\mu_k} a_{n_1}^{\mu_1\dagger}\cdots a_{n_k}^{\mu_k\dagger}|\Phi_t\rangle \,, \tag{2.17}$$

where β is an unknown coefficient satisfying

$$\beta_{n_1\cdots n_k}^* = (-1)^k(-1)^{\sum_{i=1}^k n_i}\beta_{n_1\cdots n_k} \,, \tag{2.18}$$

which is due to the hermiticity constraint of $\Phi^{(k)}$ (see appendix A). Substituting (2.17) into (2.14), we obtain equations determining β. The detailed calculation using the oscillator expression of the three-string vertex is presented in appendix A. Although the assumed state (2.17) has fixed number k of creation operators $a^\dagger$ acting on Φ_t, there emerge on the

JHEP09(2002)027

r.h.s. of (2.14) states with fewer number of $a^\dagger$ acting on Φ_{t} besides those with k $a^\dagger$s. In order to eliminate these unwanted terms, we impose the following transverse and traceless conditions on β:[2]

$$p_{\mu_1}\beta^{\mu_1\cdots\mu_k} = 0\,, \tag{2.19}$$

$$\beta_{\mu_1}{}^{\mu_1\cdots\mu_{k-1}} = 0\,. \tag{2.20}$$

Then the equation for the coefficient β is given by

$$\beta_{m_1\ldots m_k} - 2^{-p^2}\Big((\rho_-)_{m_1n_1}\cdots(\rho_-)_{m_kn_k} + (\rho_+)_{m_1n_1}\cdots(\rho_+)_{m_kn_k}\Big)\beta_{n_1\ldots n_k} = 0\,, \tag{2.21}$$

with

$$\rho_\pm = \frac{TM_\pm + M_\mp}{(1+T)(1-M_0)}\,. \tag{2.22}$$

The matrices $\rho_\pm$ are projection operators [11] satisfying

$$(\rho_\pm)^2 = \rho_\pm, \quad \rho_+\rho_- = \rho_-\rho_+ = 0, \quad \rho_+ + \rho_- = 1\,. \tag{2.23}$$

Equations (2.21) and (2.23) lead to a disappointing result that our states (2.17) can represent only massless states. Namely, multiplying (2.21) by $\rho_{s_1}\otimes\rho_{s_2}\otimes\cdots\otimes\rho_{s_k}$ with $s_i = +$ or $-$, we find that the equations for the purely ρ_+ component $(\rho_+\otimes\cdots\otimes\rho_+)\beta$ and the purely ρ_- one $(\rho_-\otimes\cdots\otimes\rho_-)\beta$ are reduced to

$$\big(1-2^{-p^2}\big)(\rho_\pm\otimes\cdots\otimes\rho_\pm)\beta = 0\,, \tag{2.24}$$

implying that they are massless states. On the other hand, (2.21) tells that the mixed components, for example, $(\rho_+\otimes\rho_-\otimes\cdots\otimes\rho_-)\beta$, are equal to zero. In the case of vector state with $k=1$, eq. (2.21) with $\rho_+ + \rho_- = 1$ substituted reproduces the result of [5] that this is a massless state and the coefficient β_n^μ is arbitrary.

The above result is inevitable so long as the basic equations (2.23) are valid. However, there is a subtle point concerning the eigenvalues of $\rho_\pm$. Recall that the eigenvalue problem of the Neumann coefficient matrices M_0 and $M_1 \equiv M_+ - M_-$ has been solved in [12]. They found that these matrices are expressed in terms of a single matrix K_1 which is the matrix representation of the Virasoro algebra $L_1 + L_{-1}$, and the eigenvalue problem of M_α is reduced to that of K_1. Let $\boldsymbol{f}^{(\kappa)}$ be the eigenvector of K_1 corresponding to the eigenvalue κ:

$$K_1\boldsymbol{f}^{(\kappa)} = \kappa\boldsymbol{f}^{(\kappa)}\,. \tag{2.25}$$

The distribution of κ is uniform and extending from $-\infty$ to ∞. This eigenvector $\boldsymbol{f}^{(\kappa)}$ is at the same time that of M_0, M_1, T and hence $\rho_\pm$. In particular we have

$$\rho_\pm\boldsymbol{f}^{(\kappa)} = \theta(\pm\kappa)\boldsymbol{f}^{(\kappa)}\,, \tag{2.26}$$

[2] Due to these two conditions, (2.19) and (2.20), our construction of massive modes is restricted only to the highest spin states at a given mass level.

where $\theta(\kappa)$ is the step function

$$\theta(\kappa) = \begin{cases} 1 & (\kappa > 0) \\ 0 & (\kappa < 0) \end{cases} . \tag{2.27}$$

The subtle point is the eigenvalue of $\rho_\pm$ at $\kappa = 0$. In fact, it has been known that there is an eigenvector $\boldsymbol{f}^{(\kappa=0)}$, which is twist-odd, $C\boldsymbol{f}^{(0)} = -\boldsymbol{f}^{(0)}$. However, the eigenvalue $\theta(\kappa = 0)$ of $\rho_\pm$ is indefinite.

If we are allowed to set $\theta(0) = 1/2$ in (2.26), which would look most plausible, the fluctuation (2.17) represents a massive state at the expected mass level,

$$p^2 = 1 - k \,, \tag{2.28}$$

by adopting either of the following two choices of $\beta^{\mu_1 \cdots \mu_k}_{n_1 \cdots n_k}$ concerning its dependence on the level number indices $n_1 \cdots n_k$:

- β is the tensor product of k $\boldsymbol{f}^{(0)}$s,

$$\beta = \boldsymbol{f}^{(0)} \otimes \cdots \otimes \boldsymbol{f}^{(0)} \,. \tag{2.29}$$

- β is the tensor product of $(k-1)$ $\boldsymbol{f}^{(0)}$s and one arbitrary vector $\boldsymbol{w}$ not equal to $\boldsymbol{f}^{(0)}$,

$$\beta = \boldsymbol{f}^{(0)} \otimes \cdots \otimes \boldsymbol{f}^{(0)} \otimes \boldsymbol{w} \,. \tag{2.30}$$

In both cases we have to multiply (2.29) and (2.30) by a transverse and traceless tensor carrying the Lorentz indices, and carry out symmetrization if necessary. Quite similarly, by taking β which is a tensor product of ℓ $\boldsymbol{f}^{(0)}$s and $k - \ell$ arbitrary vectors ($\ell \leq k - 1$), we obtain a state at mass level $p^2 = -\ell$.[3]

Now we have to resolve two problems. One is whether $\rho_\pm$ really has eigenvalue 1/2. Second, even if this is the case, we have infinite degeneracy of massive as well as massless states which are apparently physical ones. We have to show that these spurious states are gauge artifacts. Analysis of these two questions is the subject of the following two sections.

3. Numerical analysis of $\rho_\pm$

As seen in the previous section, the existence of the eigenvalue 1/2 of the "projection operators" $\rho_\pm$ was essential for the construction of the massive fluctuation modes. The corresponding eigenvector is expected to be $\boldsymbol{f}^{(0)}$, the zero-mode of K_1. It is obvious that we need some regularization for studying this expectation since what we want to know is the value of the step function $\theta(\kappa)$ (2.27) at $\kappa = 0$. In the following, we shall show numerically that $\rho_\pm$ has indeed eigenvalue 1/2 by regularizing them to finite size matrices.

[3] See section 4 for precise form of these states in the case $k = 2$.

L	$\lambda^{(1)}$	$\lambda^{(2)}$
50	0.771	0.259
100	0.752	0.279
150	0.742	0.289
200	0.735	0.295
300	0.726	0.304
500	0.716	0.314
∞	0.512	0.489

Table 1: Anomalous eigenvalues of ρ_+ (3.1) for various even L. The values at $L = \infty$ have been obtained by the fitting function of the form $\sum_{k=0}^{5} c_k / (\ln L)^k$. We use the same fitting function also in other tables 2–5.

We have solved numerically the eigenvalue problem of the regularized ρ_+ obtained by replacing M_0 and M_1 in it with $L \times L$ ones. Since we have $\rho_- = C\rho_+C$, we do not need to repeat the analysis for ρ_-. The expression (2.22) of $\rho_\pm$ was in fact obtained by naively using the non-linear relations among the Neumann coefficient matrices [15, 16] upon the original expressions, which are given by (see appendix A)

$$\rho_\pm = (M_+, M_-)(1 - T\mathcal{M})^{-1} \begin{cases} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \end{cases} , \tag{3.1}$$

with

$$\mathcal{M} = \begin{pmatrix} M_0 & M_+ \\ M_- & M_0 \end{pmatrix} . \tag{3.2}$$

Since the non-linear relations no longer hold for regularized M_α and naive use of them may be dangerous near $\kappa = 0$ [17, 18, 13], we have employed the original expression (3.1) in our numerical analysis.

Tables 1–5 show the result of our calculations. Since the eigenvalue distributions are qualitatively different between even and odd L, we have carried out the analysis for each of these two cases. In the case of even L, all the eigenvalues of ρ_+ are close to either 0 or 1 except two "anomalous" ones, $\lambda^{(1)}$ and $\lambda^{(2)}$, which are given in table 1 for various even L. Though the raw values of these anomalous eigenvalues are not so close to 1/2, their values at $L = \infty$ obtained by fitting are surprisingly close to the expected value of 1/2.

Analysis of the eigenvectors of these anomalous eigenvalues is presented in tables 2 and 3. Let us denote by $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ the eigenvectors corresponding to the eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$, respectively. These eigenvectors do not have a definite twist. We define a twist-odd vector $\boldsymbol{a}^{(i)}$ ($i = 1, 2$) with components $a^{(i)}_{2n+1} = u^{(i)}_{2n+1}/f^{(0)}_{2n+1}$, and a twist-even one $\boldsymbol{b}^{(i)}$ with $b^{(i)}_{2n} = u^{(i)}_{2n}/u^{(i)}_2$. Here, $\boldsymbol{u}^{(i)}$ is normalized so that $u^{(i)}_1 = 1$, and the components of $\boldsymbol{f}^{(0)}$ are given by [12]

$$f^{(0)}_n = \begin{cases} \dfrac{(-1)^{(n-1)/2}}{\sqrt{n}} & n\text{: odd} \\ 0 & n\text{: even} \end{cases} \tag{3.3}$$

L	$a_3^{(1)}$	$a_5^{(1)}$	$a_7^{(1)}$	$a_9^{(1)}$	$a_{11}^{(1)}$	$b_4^{(1)}$	$b_6^{(1)}$	$b_8^{(1)}$	$b_{10}^{(1)}$	$b_{12}^{(1)}$
50	1.104	1.185	1.256	1.320	1.382	−1.073	1.119	−1.030	1.208	−1.257
100	1.086	1.150	1.204	1.251	1.295	−1.040	1.055	−1.067	1.079	−1.092
150	1.077	1.134	1.183	1.223	1.260	−1.026	1.029	−1.030	1.032	−1.035
200	1.072	1.125	1.168	1.205	1.239	−1.018	1.015	−1.010	1.007	−1.005
300	1.065	1.113	1.151	1.185	1.215	−1.009	0.998	−0.987	0.977	−0.970
500	1.058	1.100	1.134	1.163	1.189	−1.000	0.981	−0.964	0.949	−0.937
∞	0.999	1.003	1.008	1.012	1.008	−0.944	0.883	−0.819	0.735	−0.606

Table 2: Components of the vectors $\boldsymbol{a}^{(1)}$ and $\boldsymbol{b}^{(1)}$ for various even L.

L	$a_3^{(2)}$	$a_5^{(2)}$	$a_7^{(2)}$	$a_9^{(2)}$	$a_{11}^{(2)}$	$b_4^{(2)}$	$b_6^{(2)}$	$b_8^{(2)}$	$b_{10}^{(2)}$	$b_{12}^{(2)}$
50	1.064	1.114	1.159	1.201	1.243	−1.030	1.038	−1.047	1.058	−1.074
100	1.056	1.096	1.130	1.161	1.189	−1.014	1.006	−0.998	0.992	−0.988
150	1.052	1.089	1.112	1.146	1.170	−1.007	0.994	−0.981	0.970	−0.961
200	1.049	1.084	1.113	1.138	1.160	−1.003	0.987	−0.972	0.958	−0.947
300	1.046	1.079	1.105	1.128	1.147	−0.999	0.979	−0.960	0.944	−0.930
500	1.043	1.072	1.096	1.117	1.134	−0.994	0.970	−0.949	0.930	−0.913
∞	0.991	0.988	0.987	0.983	0.970	−0.941	0.888	−0.840	0.785	−0.704

Table 3: Components of the vectors $\boldsymbol{a}^{(2)}$ and $\boldsymbol{b}^{(2)}$ for various even L.

Components of $\boldsymbol{a}^{(i)}$ and $\boldsymbol{b}^{(i)}$ for various even L and their values at $L=\infty$ obtained by fitting are given in tables 2 and 3 for $i=1$ and 2, respectively. They strongly support that all (odd) components of $\boldsymbol{a}^{(i)}$ are equal to one and that $\boldsymbol{b}^{(1)}$ and $\boldsymbol{b}^{(2)}$ are equal to each other.[4] This implies that $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ are given as linear combinations of two vectors, $\boldsymbol{f}^{(0)}$ which is twist-odd and $\boldsymbol{b}^{(1)}(=\boldsymbol{b}^{(2)})$ which is twist-even. Therefore, in the limit $L\to\infty$, the projection operator ρ_+ has the eigenvalue $1/2$ which is doubly degenerate, and the corresponding eigenvectors are $\boldsymbol{f}^{(0)}$ and $\boldsymbol{b}^{(1)}$. The norm of $\boldsymbol{b}^{(1)}$ seems to have worse divergence than that of $\boldsymbol{f}^{(0)}$ which is logarithmically divergent. We do not know whether $\boldsymbol{b}^{(1)}$, which appears only for even L, has any relevance to the construction of fluctuation modes.

In the case of odd L, we have only one anomalous eigenvalue λ largely deviated from either 0 or 1. Table 4 shows this eigenvalue λ for various odd L and its extrapolation to $L=\infty$. As expected, the table 4 supports that $\lambda\to 1/2$ as $L\to\infty$. The eigenvector $\boldsymbol{u}$ of this eigenvalue λ does not have definite a twist for a finite L. However, its twist-even component is negligibly small compared with the twist-odd one. In fact, the norm of the twist-even part, $(1+C)\boldsymbol{u}/2$, is at most 0.6% of the norm of the whole vector $\boldsymbol{u}$.[5] Therefore, we have studied only the twist-odd vector $\boldsymbol{a}$ with components $a_{2n+1}=u_{2n+1}/f^{(0)}_{2n+1}$. The results given in table 5 again support our expectation that $\boldsymbol{u}$ is equal to $\boldsymbol{f}^{(0)}$.

[4] If we adopt the fitting by polynomials of $1/L$, we obtain better coincidence between $b^{(1)}_{2n}$ and $b^{(2)}_{2n}$ at $L=\infty$ for larger n.

[5] In the case of even L, the norms of the even and odd parts of $\boldsymbol{u}^{(i)}$, $(1\pm C)\boldsymbol{u}^{(i)}/2$, are of the same order.

L	λ
49	0.525
99	0.523
149	0.522
199	0.521
299	0.520
499	0.519
∞	0.501

Table 4: Anomalous eigenvalues of ρ_+ (3.1) for various odd L and their extrapolation to $L = \infty$.

L	a_3	a_5	a_7	a_9	a_{11}
49	1.104	1.187	1.262	1.332	1.402
99	1.084	1.147	1.201	1.249	1.293
149	1.075	1.131	1.178	1.217	1.255
199	1.069	1.121	1.163	1.200	1.232
299	1.063	1.109	1.146	1.179	1.208
499	1.056	1.097	1.123	1.158	1.182
∞	0.997	1.000	1.001	0.993	0.961

Table 5: Components of the vector $\boldsymbol{a}$ for various odd L and their extrapolation to $L = \infty$.

In summary, in both even and odd L cases, our numerical analysis of the eigenvalue problem of $\rho_\pm$ (3.1) confirms our expectation that $\rho_\pm$ has an anomalous eigenvalue 1/2 and the corresponding eigenvector is $\boldsymbol{f}^{(0)}$.

4. Gauging away the spurious states

In section 2 we have shown that we can construct massive fluctuation modes if the projection operator $\rho_\pm$ has eigenvalue 1/2. This property has been verified numerically in the last section. However, there still remains a problem: we have infinite degeneracy of massive and massless states as we saw in the last part of section 2. In this section we shall show that these spurious states can in fact be gauged away. Our argument here is an application of that given in [10] for spurious massless vector states.

For simplicity we restrict ourselves to the lowest massive state $\Phi^{(k=2)}$:

$$|\Phi^{(2)}\rangle = \beta^{\mu\nu}_{mn} a^{\mu\dagger}_m a^{\nu\dagger}_n |\Phi_{\rm t}\rangle \,, \tag{4.1}$$

where $\beta^{\mu\nu}_{mn}$ is traceless and transverse with respect to its Lorentz indices. However, the following argument can straightforwardly be extended to $\Phi^{(k)}$ with larger k. Fourier-expanding β_{mn} in terms of the eigenvector $\boldsymbol{f}^{(\kappa)}$ of K_1,

$$\beta^{\mu\nu}_{mn} = \int d\kappa \int d\lambda\, f^{(\kappa)}_m f^{(\lambda)}_n \beta^{\mu\nu}(\kappa,\lambda) \,, \tag{4.2}$$

	$\beta^{\mu\nu}(\kappa,\lambda)$	$(\text{mass})^2$
A	$\delta(\kappa)\delta(\lambda)$	1
B	$\delta(\kappa)g(\lambda)$	1
C	arbitrary smooth function	0

Table 6: Three types of (κ,λ)-dependence of $\beta^{\mu\nu}(\kappa,\lambda)$ satisfying (4.3) and the corresponding mass squared. The function $g(\kappa)$ is an arbitrary smooth function. The whole $\beta^{\mu\nu}(\kappa,\lambda)$ must be symmetric under the exchange of (μ,κ) and (ν,λ).

and substituting it into the equation for β, (2.21) with $k=2$, we obtain the following equation for $\beta^{\mu\nu}(\kappa,\lambda)$:

$$\Bigl(1-2^{-p^2}\left[\theta(-\kappa)\theta(-\lambda)+\theta(\kappa)\theta(\lambda)\right]\Bigr)\,\beta^{\mu\nu}(\kappa,\lambda)=0\,. \tag{4.3}$$

The construction of massive states given in section 2 (see the paragraph containing (2.29) and (2.30)) can be restated as follows in terms of the spectral function $\beta^{\mu\nu}(\kappa,\lambda)$. First, (4.3) implies that $\beta^{\mu\nu}(\kappa,\lambda)$ must have support only in the regions $\{\kappa\ge 0,\lambda\ge 0\}$ and $\{\kappa\le 0,\lambda\le 0\}$. Taking into account that $\theta(0)=1/2$ as we have seen in the previous section, there are three possible types of $\beta^{\mu\nu}(\kappa,\lambda)$ giving massive as well as massless states (table 6). Since the function $g(\kappa)$ for the type-B state and $\beta^{\mu\nu}(\kappa,\lambda)$ for the type-C are arbitrary except that they are smooth (and has support only in the regions stated above for $\beta^{\mu\nu}(\kappa,\lambda)$ for type-C), these two types of states are infinitely degenerate. We shall show that the type-B and C states can in fact be removed by gauge transformations of VSFT.

The gauge transformation of VSFT, (2.3), expressed in terms of the fluctuation Φ (2.11) reads $\delta_\Lambda\Phi=\mathcal{Q}_\mathrm{B}\Lambda+\Phi*\Lambda-\Lambda*\Phi$ with $\mathcal{Q}_\mathrm{B}$ defined by (2.12). We shall consider the inhomogeneous part

$$\delta^\mathrm{I}_\Lambda\Phi=\mathcal{Q}_\mathrm{B}\Lambda\,, \tag{4.4}$$

of the whole transformation, and in particular take the following type of Λ:

$$\Lambda=\Lambda^\mathrm{m}\otimes\mathcal{I}^\mathrm{g}\,, \tag{4.5}$$

where $\mathcal{I}^\mathrm{g}$ is the ghost part of the identity string field $\mathcal{I}=\mathcal{I}^\mathrm{m}\otimes\mathcal{I}^\mathrm{g}$ satisfying $\mathcal{I}^\mathrm{g}*\Psi^\mathrm{g}_\mathrm{c}=\Psi^\mathrm{g}_\mathrm{c}*\mathcal{I}^\mathrm{g}=\Psi^\mathrm{g}_\mathrm{c}$ for $\Psi^\mathrm{g}_\mathrm{c}$ of (2.6) and $\mathcal{Q}\mathcal{I}^\mathrm{g}=0$ [19, 20, 10]. For this Λ and fluctuation Φ of the factorized form (2.13), we have

$$\delta^\mathrm{I}_\Lambda\Phi^\mathrm{m}=\Psi^\mathrm{m}_\mathrm{c}*\Lambda^\mathrm{m}-\Lambda^\mathrm{m}*\Psi^\mathrm{m}_\mathrm{c}\,. \tag{4.6}$$

As the matter part Λ^m of the gauge transformation string field, we take

$$|\Lambda^\mathrm{m}\rangle=\gamma^{\mu\nu}_{mn}a^{\mu\dagger}_m a^{\nu\dagger}_n|\Phi_\mathrm{t}\rangle\,, \tag{4.7}$$

with the coefficient $\gamma^{\mu\nu}_{mn}$ being traceless and transverse with respect to μ and ν. Then, (4.6) is given by

$$\delta^\mathrm{I}_\Lambda|\Phi^\mathrm{m}\rangle=2^{-p^2}\Bigl((\rho_-)_{mp}(\rho_-)_{nq}-(\rho_+)_{mp}(\rho_+)_{nq}\Bigr)\gamma^{\mu\nu}_{pq}a^{\mu\dagger}_m a^{\nu\dagger}_n|\Phi_\mathrm{t}\rangle\,, \tag{4.8}$$

which is expressed as the following transformation on the spectral function $\beta^{\mu\nu}(\kappa,\lambda)$ of (4.2):

$$\delta^{\mathrm{I}}_{\Lambda}\beta^{\mu\nu}(\kappa,\lambda)=2^{-p^2}\Big(\theta(-\kappa)\theta(-\lambda)-\theta(\kappa)\theta(\lambda)\Big)\gamma^{\mu\nu}(\kappa,\lambda)\,, \tag{4.9}$$

where $\gamma^{\mu\nu}(\kappa,\lambda)$ is defined for $\gamma^{\mu\nu}_{mn}$ similarly to (4.2).

Equation (4.9) implies that the type-B and C states are unphysical ones which can be eliminated by the present gauge transformation. First, the type-B states are gauged away by taking $\delta(\lambda)g(\kappa)\epsilon(\kappa)$ as the (κ,λ)-dependence of $\gamma^{\mu\nu}(\kappa,\lambda)$. Here, $\epsilon(\kappa)$ is the signature function $\epsilon(\kappa)=\theta(\kappa)-\theta(-\kappa)$. Second, $\gamma^{\mu\nu}(\kappa,\lambda)$ for gauging away the type-C states is $\gamma^{\mu\nu}(\kappa,\lambda)=-2^{p^2}\left[\theta(-\kappa)\theta(-\lambda)-\theta(\kappa)\theta(\lambda)\right]\beta^{\mu\nu}(\kappa,\lambda)$. It is obvious that the type-A states cannot be removed by the present gauge transformation.

Finally in this section we shall comment on the relation between the gauge transformation used in this paper and that in [10]. Our gauge transformation string field Λ^{m} (4.7) is of different type from that used in [10]; the latter is based on the identity string field instead of the classical solution $\Psi^{\mathrm{m}}_{\mathrm{c}}$. If we have adopted Λ^{m} of the type of [10], we would have obtained (4.8) with $\rho_{\pm}$ replaced by $(1+T)\rho_{\pm}$. This gauge transformation cannot remove the type-B states which are absent for the vector case discussed in [10].

Acknowledgments

We would like to thank K. Hashimoto, Y. Imamura, E. Itou, H. Kajiura, I. Kishimoto, Y. Matsuo, S. Moriyama, T. Muramatsu, K. Ohmori, T. Takahashi and S. Teraguchi for valuable discussions and comments. The works of H. H. was supported in part by a Grant-in-Aid for Scientific Research from Ministry of Education, Culture, Sports, Science, and Technology (#12640264).

A. Derivation of (2.21)

In this appendix we outline a derivation of (2.21) from the equation of motion (2.14). First we shall mention the hermiticity constraint (2.18). We impose the following hermiticity condition on the matter part Φ^{m} of a string field of the type (2.13):

$${}_2\langle\Phi^{\mathrm{m}}|=\prod_{r=1,2}\int\frac{d^{26}p_r}{(2\pi)^{26}}\,(2\pi)^{26}\delta^{26}(p_1+p_2)\,{}_{12}\langle R^{\mathrm{m}}|\Phi^{\mathrm{m}}\rangle_1\,, \tag{A.1}$$

where the matter reflector (two-string vertex) $\langle R^{\mathrm{m}}|$ is defined by

$${}_{12}\langle R^{\mathrm{m}}|=\langle p_1|\langle p_2|\exp\Bigg(-\sum_{n\geq 1}(-1)^n a^{(1)}_n a^{(2)}_n\Bigg)\,. \tag{A.2}$$

This constraint reduces the number of degrees of freedom in Φ^{m} to half and ensures the hermiticity of the action. Equation (2.18) for β is immediately obtained by plugging the expression (2.17) into (A.1) and using that Φ_{t} itself satisfies (A.1).

The wave equation (2.14) for the matter fluctuation Φ^{m} is rewritten in the oscillator representation as

$$|\Phi^{\mathrm{m}}\rangle_3 - {}_1\langle\Psi_{\mathrm{c}}^{\mathrm{m}}|_2\langle\Phi^{\mathrm{m}}|V^{\mathrm{m}}\rangle_{123} - {}_1\langle\Phi^{\mathrm{m}}|_2\langle\Psi_{\mathrm{c}}^{\mathrm{m}}|V^{\mathrm{m}}\rangle_{123} = 0\,, \tag{A.3}$$

where we have omitted the integrations $\left(\prod_{r=1}^{3}\int d^{26}p_r/(2\pi)^{26}\right)(2\pi)^{26}\delta^{26}(p_1+p_2+p_3)$ in the second and third terms. Let us consider the second term of (A.3) with $\Phi^{(k)}$ (2.17) substituted for Φ^{m}, ${}_1\langle\Psi_{\mathrm{c}}^{\mathrm{m}}|_2\langle\Phi^{(k)}|V^{\mathrm{m}}\rangle_{123}$. The basic formula for this calculation is obtained from the well-known formula valid for any bosonic oscillators satisfying $[a_i, a_j^\dagger] = \delta_{ij}$:

$$\begin{aligned}\langle 0|\exp\left(-\frac{1}{2}a_iA_{ij}a_j - K_ia_i\right)\exp\left(-\frac{1}{2}a_i^\dagger B_{ij}a_j^\dagger - J_ia_i^\dagger\right)|0\rangle \\ = [\det(1-AB)]^{-1/2}\exp\left(-\frac{1}{2}JPAJ - \frac{1}{2}KBPK + JPK\right),\end{aligned} \tag{A.4}$$

with $P = (1-AB)^{-1}$. Letting $(-\partial/\partial K_{i_1})\cdots(-\partial/\partial K_{i_k})$ act on (A.4), we get

$$\begin{aligned}&\langle 0|a_{i_1}\cdots a_{i_k}\exp\left(-\frac{1}{2}a_iA_{ij}a_j - K_ia_i\right)\exp\left(-\frac{1}{2}a_i^\dagger B_{ij}a_j^\dagger - J_ia_i^\dagger\right)|0\rangle \\ &\quad = [\det(1-AB)]^{-\frac{1}{2}}\times \\ &\qquad\times\left\{\prod_{a=1}^{k}(KBP-JP)_{i_a} + \sum_{a>b}(BP)_{i_ai_b}\prod_{c\neq a,b}(KBP-JP)_{i_c} + \ldots\right\}\times \\ &\qquad\times\exp\left(-\frac{1}{2}JPAJ - \frac{1}{2}KBPK + JPK\right),\end{aligned} \tag{A.5}$$

where we have omitted terms with more than one $(BP)_{i_ai_b}$ factors. In our applications of (A.5), the index i represents the level number n, the Lorentz index μ and the string index $r = 1, 2$. Contracting (A.5) with $(\beta^{\mu_1\cdots\mu_k}_{n_1\cdots n_k})^*$, the terms containing the $(BP)_{i_ai_b}$ factors drop out due to the traceless condition (2.20), while due to the transverse condition the $(KBP-JP)_i$ factor with $i = (n, \mu, r = 2)$ contributes only $-a_m^{\mu(3)\dagger}(\rho_- C)_{mn}$ with ρ_- defined by (3.1). Therefore, we have

$$\begin{aligned}{}_1\langle\Psi_{\mathrm{c}}^{\mathrm{m}}|_2\langle\Phi^{(k)}|V^{\mathrm{m}}\rangle_{123} &= 2^{-p^2}(-1)^k a_{m_1}^{\mu_1(3)\dagger}\cdots a_{m_k}^{\mu_k(3)\dagger}(\rho_- C)_{m_1n_1}\cdots(\rho_- C)_{m_kn_k}(\beta^{\mu_1\cdots\mu_k}_{n_1\cdots n_k})^*|\Phi_{\mathrm{t}}\rangle \\ &= 2^{-p^2}a_{m_1}^{\mu_1(3)\dagger}\cdots a_{m_k}^{\mu_k(3)\dagger}(\rho_-)_{m_1n_1}\cdots(\rho_-)_{m_kn_k}\beta^{\mu_1\cdots\mu_k}_{n_1\cdots n_k}|\Phi_{\mathrm{t}}\rangle\,,\end{aligned} \tag{A.6}$$

where we have used $\Psi_{\mathrm{c}}^{\mathrm{m}} * \Phi_{\mathrm{t}} = 2^{-p^2}\Phi_{\mathrm{t}}$ and the hermiticity (2.18). This gives the ρ_- term of (2.21). Derivation of the ρ_+ term of (2.21) from the last term of (A.3) is quite similar.

References

[1] L. Rastelli, A. Sen and B. Zwiebach, *String field theory around the tachyon vacuum*, hep-th/0012251.

[2] L. Rastelli, A. Sen and B. Zwiebach, *Classical solutions in string field theory around the tachyon vacuum*, hep-th/0102112.

[3] L. Rastelli, A. Sen and B. Zwiebach, *Boundary CFT construction of D-branes in vacuum string field theory*, *J. High Energy Phys.* **11** (2001) 045 [hep-th/0105168].

[4] L. Rastelli, A. Sen and B. Zwiebach, *Vacuum string field theory*, hep-th/0106010.

[5] H. Hata and T. Kawano, *Open string states around a classical solution in vacuum string field theory*, *J. High Energy Phys.* **11** (2001) 038 [hep-th/0108150].

[6] L. Rastelli, A. Sen and B. Zwiebach, *A note on a proposal for the tachyon state in vacuum string field theory*, *J. High Energy Phys.* **02** (2002) 034 [hep-th/0111153].

[7] R. Rashkov and K.S. Viswanathan, *A note on the tachyon state in vacuum string field theory*, hep-th/0112202.

[8] R. Rashkov and K.S. Viswanathan, *A proposal for the vector state in vacuum string field theory*, hep-th/0201229.

[9] Y. Okawa, *Open string states and D-brane tension from vacuum string field theory*, *J. High Energy Phys.* **07** (2002) 003 [hep-th/0204012].

[10] Y. Imamura, *Gauge transformations on a D-brane in vacuum string field theory*, *J. High Energy Phys.* **07** (2002) 042 [hep-th/0204031].

[11] L. Rastelli, A. Sen and B. Zwiebach, *Half strings, projectors and multiple D-branes in vacuum string field theory*, *J. High Energy Phys.* **11** (2001) 035 [hep-th/0105058].

[12] L. Rastelli, A. Sen and B. Zwiebach, *Star algebra spectroscopy*, *J. High Energy Phys.* **03** (2002) 029 [hep-th/0111281].

[13] H. Hata and S. Moriyama, *Reexamining classical solution and tachyon mode in vacuum string field theory*, hep-th/0206208.

[14] V.A. Kostelecky and R. Potting, *Analytical construction of a nonperturbative vacuum for the open bosonic string*, *Phys. Rev.* **D 63** (2001) 046007 [hep-th/0008252].

[15] D.J. Gross and A. Jevicki, *Operator formulation of interacting string field theory*, *Nucl. Phys.* **B 283** (1987) 1.

[16] D.J. Gross and A. Jevicki, *Operator formulation of interacting string field theory 2*, *Nucl. Phys.* **B 287** (1987) 225.

[17] H. Hata and S. Moriyama, *Observables as twist anomaly in vacuum string field theory*, *J. High Energy Phys.* **01** (2002) 042 [hep-th/0111034].

[18] H. Hata, S. Moriyama and S. Teraguchi, *Exact results on twist anomaly*, *J. High Energy Phys.* **02** (2002) 036 [hep-th/0201177].

[19] I. Kishimoto, *Some properties of string field algebra*, *J. High Energy Phys.* **12** (2001) 007 [hep-th/0110124].

[20] I. Kishimoto and K. Ohmori, *CFT description of identity string field: toward derivation of the VSFT action*, *J. High Energy Phys.* **05** (2002) 036 [hep-th/0112169].